Degenerations of Mixed Hodge Structure

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1 Introduction

In this paper we extend Schmid's Nilpotent Orbit Theorem to admissible variations of graded-polarized mixed Hodge structure

$$\mathcal{V} \to \Delta^*$$

and derive analogs of the harmonic metric equations for variations of graded-polarized mixed Hodge structure. The original motivation for the study of such variations rests upon the following observation [4]: Let $f : Z \to S$ be a surjective, quasi-projective morphism. Then, the sheaf

$$\mathcal{V} = R^k_{f*}(\mathbb{C})$$

restricts to a variation of graded-polarized mixed Hodge structure over some Zariski-dense open subset of S. More recently, such variations have been shown to arise in connection with the study of the monodromy representations of smooth projective varieties [8] as well as certain aspects of mirror symmetry [6].

The basic problem of identifying a good class of abstract variations of graded-polarized mixed Hodge structure for which one could expect to obtain analogs of Schmid's orbit theorems was posed by Deligne in [4]. The accepted answer to this question was provided by Steenbrink and Zucker in [16], wherein they introduced the category of *admissible* variations of graded-polarized mixed Hodge structure, and proved that every geometric variation $\mathcal{V}$ defined over a smooth, quasi-projective curve X is admissible, and moreover the cohomology $H^k(X, \mathcal{V})$ of such a curve X with coefficients in an admissible variation $\mathcal{V} \to X$ carries a functorial mixed Hodge structure. The original question of developing analogs of Schmid's orbit theorems for such variations has however remained largely unresolved.

The general outline of the paper is as follows: In §2, we review the basic properties of the classifying spaces of graded-polarized mixed Hodge structures

$$\mathcal{M} = \mathcal{M}(W, \mathcal{S}, \{h^{p,q}\})$$

constructed in [12] and recall how the isomorphism class of a variation of graded-polarized mixed Hodge structure $\mathcal{V} \to S$ may be recovered from the knowledge of its monodromy representation:

$$\rho : \pi_1(S, s_0) \to GL(\mathcal{V}_{s_0}), \qquad \text{Image}(\rho) = \Gamma$$

and its period map

$$\varphi : S \to \mathcal{M}/\Gamma$$

Following [9], we then construct a natural hermitian metric h on $\mathcal{M}$ which is invariant under the action of the Lie group

$$G_{\mathbb{R}} = \{\, g \in GL(V_{\mathbb{R}})^W \mid Gr(g) \in Aut(\mathcal{S}, \mathbb{R}) \,\}$$

Remark. *The Lie group $G_{\mathbb{R}}$ defined above only acts transitively upon the real points of $\mathcal{M}$ (i.e. the points $F \in \mathcal{M}$ for which the corresponding mixed Hodge structure (F, W) is split over $\mathbb{R}$).*

In §3, we derive analogs of the harmonic metric equations for filtered vector bundles and determine necessary and sufficient conditions for such a filtered harmonic metric to underlie a complex variation of graded-polarized mixed Hodge structure.

In §4, we recall the notion of an admissible variation of graded polarized mixed Hodge structure and present a result of P. Deligne [5] which shows how to construct a distinguished sl_2-representation from the limiting data of an admissible variation $\mathcal{V} \to \Delta^*$.

Making use of the material of §2 and §4, we then proceed in §5 to prove an analog of Schmid's Nilpotent Orbit Theorem for admissible variations of graded-polarized mixed Hodge structure $\mathcal{V} \to \Delta^*$ which gives distance estimates identical to those of the pure case. Namely, once the period mapping of such a variation is lifted to a map of the upper half-plane into $\mathcal{M}$, the Hodge filtration of the given variation and that of the associated nilpotent orbit satisfy an estimate of the form

$$d_{\mathcal{M}}(F(z), e^{zN}.F_{\infty}) \leq K\, \mathrm{Im}(z)^{\beta} \exp(-2\pi\, \mathrm{Im}(z))$$

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2 Preliminary Remarks

In this section we review some background material from [16] and [12].

Definition 2.1 *Let S be a complex manifold. Then, a variation of graded-polarized mixed Hodge structure $\mathcal{V} \to S$ consists of a $\mathbb{Q}$-local system $\mathcal{V}_{\mathbb{Q}}$ over S equipped with:*

(1) A rational, increasing weight filtration $0 \subseteq \cdots \mathcal{W}_k \subseteq \mathcal{W}_{k+1} \subseteq \cdots \subseteq \mathcal{V}_{\mathbb{C}}$ of $\mathcal{V}_{\mathbb{C}} = \mathcal{V}_{\mathbb{Q}} \otimes \mathbb{C}$.

(2) A decreasing Hodge filtration $0 \subseteq \cdots \mathcal{F}^p \subseteq \mathcal{F}^{p-1} \subseteq \cdots \subseteq \mathcal{V}$ of $\mathcal{V} = \mathcal{V}_{\mathbb{C}} \otimes \mathcal{O}_S$ by holomorphic subbundles.

(3) A collection of rational, non-degenerate bilinear forms

$$\mathcal{S}_k : Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}) \otimes Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}) \to \mathbb{Q}$$

of alternating parity $(-1)^k$.

satisfying the following mutual compatibility conditions:

(a) Relative to the Gauss–Manin connection of $\mathcal{V}$:

$$\nabla \mathcal{F}^p \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}$$

(b) For each index k, the triple $(Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}), \mathcal{F}\, Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}), \mathcal{S}_k)$ defines a variation of pure, polarized Hodge structure of weight k.

As discussed in [12], the data of such a variation $\mathcal{V} \to S$ may be effectively encoded into its monodromy representation

$$\rho : \pi_1(S, s_0) \to GL(\mathcal{V}_{s_0}), \qquad \mathrm{Image}(\rho) = \Gamma \tag{2.2}$$

and its period map

$$\phi : S \to \mathcal{M}/\Gamma \tag{2.3}$$

To obtain such a reformulation, observe that it suffices to consider a variation $\mathcal{V}$ defined over a simply connected base space S. Trivialization of $\mathcal{V}$ relative to a fixed reference fiber $V = \mathcal{V}_{s_0}$ via parallel translation will then determine the following data:

(1) A rational structure $V_{\mathbb{Q}}$ on V.

(2) A rational weight filtration W of V.

(3) A variable Hodge filtration $F(s)$ of V.

(4) A collection of rational, non-degenerate bilinear forms

$$\mathcal{S}_k : Gr^W_k \otimes Gr^W_k \to \mathbb{C}$$

of alternating parity $(-1)^k$.

subject to the restrictions:

(a) The Hodge filtration $F(s)$ is holomorphic and horizontal, i.e.

$$\frac{\partial}{\partial \bar{s}_j} F^p(s) \subseteq F^p(s), \qquad \frac{\partial}{\partial s_j} F^p(s) \subseteq F^{p-1}(s) \tag{2.4}$$

relative to any choice of holomorphic coordinates $(s_1, \dots, s_n)$ on S.

(b) Each pair $(F(s), W)$ is a mixed Hodge structure, graded-polarized by the bilinear forms $\{\mathcal{S}_k\}$.

Conversely, the data listed in items (1)–(4) together with the restrictions (a) and (b) suffice to determine a VGPMHS over a simply connected base.

To extract from these properties an appropriate classifying space of graded-polarized mixed Hodge structures, observe that by virtue of conditions (a) and (b), the graded Hodge numbers $h^{p,q}$ of $\mathcal{V}$ are constant. Consequently, the filtration $F(s)$ must assume values in the set

$$\mathcal{M} = \mathcal{M}(W, \mathcal{S}, h^{p,q})$$

consisting of all decreasing filtrations F of V such that

- (F, W) is a mixed Hodge structure, graded-polarized by $\mathcal{S}$.
- $\dim_{\mathbb{C}} F^p Gr^W_k = \sum_{r \geq p} h^{r,k-r}$.

To obtain a complex structure on $\mathcal{M}$, one simply exhibits $\mathcal{M}$ as an open subset of an appropriate "compact dual" $\check{\mathcal{M}}$. More precisely, one starts with the flag variety $\check{\mathcal{F}}$ consisting of all decreasing filtrations F of V such that

$$\dim F^p = f^p, \qquad f^p = \sum_{r \geq p, s} h^{r,s}$$

To take account of the weight filtration W, one then defines $\check{\mathcal{F}}(W)$ to be the submanifold of $\check{\mathcal{F}}$ consisting of all filtrations $F \in \check{\mathcal{F}}$ which have the additional property that

$$\dim F^p Gr^W_k = \sum_{r \geq p} h^{r,k-r}$$

As in the pure case, the appropriate "compact dual" $\check{\mathcal{M}} \subseteq \check{\mathcal{F}}(W)$ is the submanifold of $\check{\mathcal{F}}(W)$ consisting of all filtrations $F \in \check{\mathcal{F}}(W)$ which satisfy Riemann's first bilinear relation with respect to the graded-polarizations $\mathcal{S}$. In particular, as shown in [12], $\check{\mathcal{M}}$ contains the classifying space $\mathcal{M}$ as a dense open subset.

In order to state our next result, we recall that [3] each choice of a mixed Hodge structure (F, W) on a complex vector space $V = \mathcal{V}_{\mathbb{R}} \otimes \mathbb{C}$ determines a unique, functorial bigrading

$$V = \bigoplus_{p,q} I^{p,q}$$

with the following three properties:

(1) For each index p, $F^p = \oplus_{a \geq p} I^{a,b}$.

(2) For each index k, $W_k = \oplus_{a+b \leq k} I^{a,b}$.

(3) For each bi-index (p, q), $\bar{I}^{p,q} = I^{q,p} \mod \oplus_{r<q,s<p} I^{r,s}$.

In analogy with the pure case, I shall call this decomposition the Deligne–Hodge decomposition of V. In particular, as discussed in [12], the pointwise application of this construction to a variation of graded-polarized mixed Hodge structure $\mathcal{V} \to S$ determines a smooth decomposition

$$\mathcal{V} = \bigoplus_{p,q} \mathcal{I}^{p,q}$$

of $\mathcal{V}$ into a sum of C^∞-subbundles.

Theorem. (Kaplan, [9]) *Let $\mathcal{M}_{\mathbb{R}}$ denote the set of all filtrations $F \in \mathcal{M}$ for which the corresponding mixed Hodge structure (F, W) is split over $\mathbb{R}$, (i.e. $\overline{I^{p,q}} = I^{q,p}$) and $G_{\mathbb{R}} = \{\, g \in GL(V_{\mathbb{R}})^W \mid Gr(g) \in Aut(\mathcal{S}, \mathbb{R}) \,\}$. Then,*

- *The group $G_{\mathbb{R}}$ acts transitively on $\mathcal{M}_{\mathbb{R}}$.*

- *The group $G_{\mathbb{C}} = \{\, g \in GL(V)^W \mid Gr(g) \in Aut(\mathcal{S}, \mathbb{C}) \,\}$ acts transitively on $\check{\mathcal{M}}$.*

- *The intermediate group $G = \{\, g \in GL(V)^W \mid Gr(g) \in Aut(\mathcal{S}, \mathbb{R}) \,\}$ acts transitively on $\mathcal{M}$.*

Remark. *By virtue of functoriality, each graded-polarized mixed Hodge structure (F, W) determines a natural mixed Hodge structure on the Lie group $\mathfrak{g} = Lie(G_{\mathbb{C}})$ via the bigrading $\mathfrak{g}^{r,s}_{(F,W)} = \{\, a \in \mathfrak{g} \mid \alpha : I^{p,q}_{(F,W)} \to I^{p+r,q+s}_{(F,W)} \,\}$.*

To measure distances in $\mathcal{M}$, we shall now describe the construction of a natural hermitian metric h carried by the classifying space $\mathcal{M}$. For the most part, our presentation follows the unpublished notes [9]. To motivate this construction, recall that in the pure case, there are essentially two equivalent ways of constructing a natural hermitian metric on the classifying spaces $\mathcal{D}$:

(1) Use the fact that $G_{\mathbb{R}}$ acts transitively on $\mathcal{D}$ with compact isotopy to define a $G_{\mathbb{R}}$-invariant metric on $\mathcal{D}$.

(2) Use the flag manifold structure of $\mathcal{D}$ and the Hodge metric

$$h_F(u, v) = \mathcal{S}(C_F u, \bar{v})$$

to induce a hermitian metric on $\mathcal{D}$.

The problem with using the first approach in the mixed case is that

- Although $G^F_{\mathbb{R}}$ is still compact, the group $G_{\mathbb{R}}$ does in general act transitively upon $\mathcal{M}$.

- Although G does act transitively upon $\mathcal{M}$, this action does not in general have compact isotopy.

Thus, in order to construct a natural hermitian metric on $\mathcal{M}$, we abandon the first approach and try instead to construct a natural generalization of the Hodge metric.

Lemma. (Mixed Hodge Metric, [9]) *Let $(F, W, \mathcal{S})$ be a graded-polarized mixed Hodge structure with underlying vector space $V = V_{\mathbb{R}} \otimes \mathbb{C}$. Then, there exists a unique, positive-definite hermitian inner product*

$$h_F = h_{(F,W,\mathcal{S})}$$

on V with the following two properties:

(a) *The bigrading* $V = \bigoplus_{p,q} I^{p,q}_{(F,W)}$ *is orthogonal with respect to* h_F.

(b) *If* u *and* v *are elements of* $I^{p,q}$, *then* $h_F(u,v) = i^{p-q}\mathcal{S}_{p+q}([u],[\bar{v}])$.

Theorem 2.5 *The mixed Hodge metric* h *defined above determines a natural hermitian metric on the classifying spaces of graded-polarized mixed Hodge structure*

$$\mathcal{M} = \mathcal{M}(W, \mathcal{S}, h^{p,q})$$

which is invariant under the action of $G_{\mathbb{R}} : \mathcal{M} \to \mathcal{M}$.

Proof. Let F be a point of the classifying space $\mathcal{M}$ and define

$$q_F = \bigoplus_{r<0, r+s\leq 0} \mathfrak{g}^{r,s}_{(F,W)} \subseteq Lie(G_{\mathbb{C}})$$

Then, as discussed in [12], the subalgebra q_F is a vector space complement to $Lie(G^F_{\mathbb{C}})$ in $Lie(G_{\mathbb{C}})$, and hence the map $\exp : u \in q_F \mapsto e^u.F \in \mathcal{M}$ restricts to a biholomorphism from some neighborhood of zero in q_F onto some neighborhood of F in $\mathcal{M}$. Consequently, we may introduce a hermitian metric on $T_F(\mathcal{M})$ by first identifying $T_F(\mathcal{M})$ with $q_F \cong T_0(q_F)$ via the differential $\exp_* : T_0(q_F) \to T_F(\mathcal{M})$ and then applying the rule:

$$h_F(\alpha, \beta) = Tr(\alpha\beta^*) \tag{2.6}$$

To see that $G_{\mathbb{R}}$ act by isometries with respect to h, one simply computes using (2.6). $\square$

Example 2.7 *Let* $\mathcal{M} = \mathcal{M}(W, \mathcal{S}, h^{p,q})$ *be the classifying space of graded-polarized mixed Hodge structure defined by the following data*

(1) Rational structure: $V_{\mathbb{Q}} = span_{\mathbb{Q}}(e_0, e_2)$.

(2) Hodge numbers: $h^{1,1} = 1$, $h^{0,0} = 1$.

(3) Weight filtration:

$$0 = W_{-1} \subset W_0 = span(e_0) = W_1 \subset W_2 = V$$

(4) Graded Polarizations: $\mathcal{S}_{2j}([e_j],[e_j]) = 1$.

Then, $\mathcal{M}$ is isomorphic to $\mathbb{C}$ via the map

$$\lambda \in \mathbb{C} \mapsto F(\lambda) \in \mathcal{M}, \qquad F^1(\lambda) = span(e_2 + \lambda e_0)$$

Moreover, relative to the mixed Hodge metric, the following frame is both holomorphic and unitary:

$$v_1(\lambda) = e_0, \qquad v_2(\lambda) = e_2 + \lambda e_0$$

Consequently, the classifying space $\mathcal{M}$ is necessarily flat (relative to the mixed Hodge metric).

Remark. *This example also shows that (in contrast with the pure case) the period map $\phi : \Delta^* \to \mathcal{M}/\Gamma$ of an abstract variation of graded-polarized mixed Hodge structure $\mathcal{V} \to \Delta^*$ may have an irregular singularity at $s = 0$. To wit: Set $\Gamma = \{1\}$ and define*

$$\phi(s) = \exp(\frac{1}{s}) : \Delta^* \to \mathcal{M} \tag{2.8}$$

where $\mathcal{M} \cong \mathbb{C}$ is the classifying space of Example (2.7).

As a prelude to §4, we now recall some background material from [16] and record some auxiliary results by the relative weight filtration ${}^rW$ and its relationship with the finite dimensional representations of $sl_2(\mathbb{C})$. To this end, we shall let V denote a finite dimensional $\mathbb{C}$-vector space and define $E_\alpha(Y)$ to be the α-eigenspace of a semisimple endomorphism $Y : V \to V$.

Theorem 2.9 *Given a nilpotent endomorphism $N : V \to V$, there exists a unique* monodromy weight filtration

$$0 \subset W(N)_{-k} \subseteq W(N)_{1-k} \subseteq \cdots \subseteq W(N)_{k-1} \subseteq W(N)_k = V$$

of V such that:

- *$N : W(N)_j \to W(N)_{j-2}$ for each index j.*
- *The induced maps $N^j : Gr^{W(N)}_j \to Gr^{W(N)}_{-j}$ are isomorphisms.*

Example 2.10 *Let ρ be a finite dimensional representation of $sl_2(\mathbb{C})$ and*

$$N_{\pm} = \rho(n_{\pm}), \qquad Y = \rho(y)$$

denote the images of the standard generators (n_-, y, n_+) of $sl_2(\mathbb{C})$. Then, by virtue of the semisimplicity of $sl_2(\mathbb{C})$ and the commutator relations

$$[Y, N_{\pm}] = \pm 2N_{\pm}, \qquad [N_+, N_-] = Y$$

it follows that:

$$W(N_-)_k = \bigoplus_{j \leq k} E_j(Y) \tag{2.11}$$

To obtain a converse of the construction given in the preceding example, recall that a grading Y of an increasing filtration W of a finite dimensional vector space V may be viewed as a semisimple element of $End(V)$ such that

$$W_k = E_k(Y) \oplus W_{k-1}$$

for each index $k \in \mathbb{Z}$. In particular, each mixed Hodge structure (F, W) defines a functorial grading $Y_{(F,W)}$ of the underlying weight filtration W via the rule

$$v \in I^{p,q}_{(F,W)} \implies Y_{(F,W)}(v) = (p+q)v$$

Moreover, given any increasing filtration W of a complex vector space V, the set $\mathcal{Y}(W) \subset End(V)$ consisting of all gradings Y of W is an affine space upon which the nilpotent Lie algebra

$$Lie_{-1} = \{\, \alpha \in End(V) \mid \alpha : W_k \to W_{k-1} \;\; \forall\, k \,\}$$

acts transitively [3].

Theorem 2.12 *Let N be a fixed, non-trivial, nilpotent endomorphism of V, and*

$$(n_-, y, n_+)$$

denote the standard generators of $sl_2(\mathbb{C})$. Then, there exists a bijective correspondence between:

(a) The set S of all gradings H of the monodromy weight filtration $W(N)$ for which

$$[H, N] = -2N$$

(b) The set S' of all representations

$$\rho : sl_2(\mathbb{C}) \to End(V)$$

such that $\rho(n_-) = N$.

Proof. The construction of Example (2.10) defines a map $f : S' \to S$ by virtue of (2.11) and the standard commutator relations of $sl_2(\mathbb{C})$. To obtain an map $g : S \to S'$ such that

$$f \circ g = Id, \qquad g \circ f = Id$$

one simply considers how the Jordan blocks of N interact with the grading H of $W(N)$. □

Definition 2.13 *Given an increasing filtration W of V and an integer $\ell \in \mathbb{Z}$ the corresponding shifted object $W[\ell]$ is the increasing filtration of V defined by the rule:*

$$W[\ell]_j = W_{j+\ell}$$

Theorem 2.14 *Let W be an increasing filtration of V. Then, given a nilpotent endomorphism $N : V \to V$ which preserves W, there exists at most one increasing filtration of V*

$${}^rW = {}^rW(N, W)$$

with the following two properties:

- *For each index j, $N : {}^rW_j \to {}^rW_{j-2}$*
- *For each index k, ${}^rW$ induces on Gr^W_k the corresponding shifted monodromy weight filtration*

$$W(N : Gr^W_k \to Gr^W_k)[-k]$$

Following [16], we shall call ${}^rW(N, W)$ the relative weight filtration of W and N. To relate this filtration with the finite dimensional representations of $sl_2(\mathbb{C})$, we may proceed as follows:

Theorem 2.15 *Let ${}^rY$ be a grading of ${}^rW = {}^rW(N, W)$ such that*

- *${}^rY$ preserves W.*

- $[{}^rY, N] = -2N$.

Then, each choice of grading Y of W which preserves ${}^rW$ determines a corresponding representation $\rho : sl_2(\mathbb{C}) \to End(V)$ such that

$$\rho(n_-) = N_0, \qquad \rho(y) = {}^rY - Y$$

where

$$N = N_0 + N_{-1} + N_{-2} + \cdots \tag{2.16}$$

denotes the decomposition of N relative to the eigenvalues of $ad\,Y$.

Proof. By virtue of the definition of the relative weight filtration and the mutual compatibility of ${}^rY$ with Y, it follows that the induced map

$$H = Gr({}^rY - Y) : Gr^W \to Gr^W$$

grades the monodromy weight filtration $W(N : Gr^W \to Gr^W)$. Moreover, by hypothesis

$$[H, Gr(N)] = -2Gr(N)$$

Thus, application of Theorem (2.12) defines a collection of representations

$$\rho_k : sl_2(\mathbb{C}) \to Gr_k^W$$

which we may then lift to the desired representation $\rho : sl_2(\mathbb{C}) \to End(V)$ via the grading Y.

To see that $\rho(n_-) = N_0$ and $\rho(y) = {}^rY - Y$, observe that if

$$\alpha = \alpha_0 + \alpha_{-1} + \alpha_{-2} + \cdots$$

is the decomposition of an endomorphism $\alpha \in End(V)^W$ according to the eigenvalues of $ad\,Y$ then the lift of

$$Gr(\alpha) : Gr^W \to Gr^W$$

with respect to the induced isomorphism $Y : Gr^W \to V$ is exactly α_0. Thus $\rho(n_-) = N_0$. Likewise, $\rho(y) = {}^rY - Y$ since the mutual compatibility of the gradings ${}^rY$ and Y implies that ${}^rY$ and Y may be simultaneously diagonalized. □

To close this section, we now recall the following definition from [16]:

Definition 2.17 *A variation $\mathcal{V} \to \Delta^*$ of graded-polarized mixed Hodge structure with unipotent monodromy is said to be admissible provided:*

(i) *The limiting Hodge filtration F_∞ of $\mathcal{V}$ exists.*

(ii) *The relative weight filtration ${}^rW = {}^rW(N, W)$ of the monodromy logarithm N and the weight filtration W of $\mathcal{V}$ exists.*

The precise meaning of condition (i) is a follows: Let

$$\varphi : \Delta^* \to \mathcal{M}/\Gamma$$

be the period map of a variation of graded-polarized mixed Hodge structure with unipotent monodromy. Then, as in the pure case, $\varphi(s)$ may be lifted to a holomorphic, horizontal map

$$F(z) : U \to \mathcal{M}$$

from the upper half-plane U into $\mathcal{M}$ which makes the following diagram commute:

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, "s=\exp(2\pi i z)"'] & \mathcal{M} \arrow[d] \\ \Delta^* \arrow[r, "\phi"] & \mathcal{M}/\Gamma \end{tikzcd}$$

In particular, on account of this diagram,

$$F(z+1) = e^N.F(z)$$

and hence $F(z)$ descends to a "untwisted period map"

$$\psi(s) : \Delta^* \to \check{\mathcal{M}}$$

In accord with [13], the limiting value of $\psi(s)$ at zero [when it exists] is then called the limiting Hodge filtration F_∞ of $\mathcal{V}$.

Remark. *To be coordinate free, the limiting Hodge filtration F_∞ constructed above should be viewed as an object attached to $T_0(\Delta)$.*

3 Higgs Fields and Harmonic Metrics

In this section we derive analogs of the harmonic metric equations for filtered vector bundles and discuss their relationship with complex variations of graded-polarized mixed Hodge structure.

Definition 3.1 *A Higgs bundle consists of a C^{∞} complex vector bundle endowed with a smooth linear map $\theta : E \to E \otimes \mathcal{E}^1$ of type $(1,0)$ and a smooth differential operator $\bar{\partial}$ of type $(0,1)$ which satisfy the following mutual compatibility condition:*

$$(\bar{\partial} + \theta)^2 = 0$$

Equivalently, by virtue of the Newlander–Nirenberg theorem, a Higgs bundle may be viewed as holomorphic vector bundle $(E, \bar{\partial})$ endowed with the choice of a holomorphic map $\theta : E \to E \otimes \Omega^1$ such that $\theta \wedge \theta = 0$.

Construction 3.2 *Let (E, ∇) be flat vector bundle. Then, each choice of a hermitian metric h determines a unique choice of differential operators δ' and δ'' of respective types $(1,0)$ and $(0,1)$ such that the corresponding connections $\delta' + \nabla''$ and $\nabla' + \delta''$ preserve the metric h. In particular, each choice of a hermitian metric h on a flat vector bundle E determines a canonical decomposition*

$$\nabla = \bar{\theta} + \bar{\partial} + \partial + \theta$$

of the underlying flat connection ∇ via the rule:

$$\bar{\theta} = \frac{1}{2}(\nabla'' - \delta''), \quad \bar{\partial} = \frac{1}{2}(\nabla'' + \delta''), \quad \partial = \frac{1}{2}(\nabla' + \delta'), \quad \theta = \frac{1}{2}(\nabla' - \delta')$$

Motivated by [14], we define a hermitian metric h on a flat vector bundle E to be *harmonic* provided the resulting operator $\bar{\partial} + \theta$ produced by Construction (3.2) is of Higgs–type [i.e. $(\bar{\partial} + \theta)^2 = 0$].

Example 3.3 *The Hodge metric h carried by a variation of pure, polarized Hodge structure $\mathcal{V} \to S$ is harmonic.*

Lemma. *Let X be a compact Kähler manifold. Then, a flat vector bundle $E \to X$ admits a harmonic metric h if and only if it is semisimple.*

In particular, since monodromy representation of a variation of graded-polarized mixed Hodge structure need not be semisimple, the underlying flat vector bundle of a VGPMHS is need not admit a harmonic metric. Accordingly, we devote the remainder of this section to deriving analogs of the harmonic metric equations for flat, filtered vector bundles.

Definition 3.4 *A filtered bundle $(E, \mathcal{W})$ consists of a C^∞ vector bundle E endowed with an increasing filtration*

$$0 \subseteq \cdots \subseteq \mathcal{W}_{k-1} \subseteq \mathcal{W}_k \subseteq \cdots \subseteq E$$

of E by C^∞ subbundles. Likewise, a flat, filtered bundle $(E, \mathcal{W}, \nabla)$ consists of a C^∞ vector bundle E endowed with an increasing filtration by flat subbundles $\mathcal{W}_k \subseteq E$.

Construction 3.5 *Each choice of a hermitian metric h on a filtered bundle $(E, \mathcal{W})$ defines a corresponding grading $\mathcal{Y}_h$ of the underlying weight filtration $\mathcal{W}$ by simply declaring $E_k(\mathcal{Y}_h)$ to be the orthogonal complement of $\mathcal{W}_{k-1}$ in $\mathcal{W}_k$ with respect to h.*

In particular, each choice of a hermitian metric h on a flat, filtered bundle $(E, \mathcal{W}, \nabla)$ determines a unique decomposition of the underlying flat connection ∇ into a sum of three terms

$$\nabla = d_h + \tau_- + \theta_- \tag{3.6}$$

such that

- d_h is a connection which preserves the grading $\mathcal{Y}_h$.
- τ_- is a nilpotent tensor field of type $(0,1)$ which maps $\mathcal{W}_k$ to $\mathcal{W}_{k-1}$ for each index k.
- θ_- is a nilpotent tensor field of type $(1,0)$ which maps $\mathcal{W}_k$ to $\mathcal{W}_{k-1}$ for each index k.

Moreover, the resulting connection d_h is automatically flat, as may be seen by expanding out the flatness condition $\nabla^2 = 0$ and taking note of the fact that $d_h : E_k(\mathcal{Y}_h) \to E_k(\mathcal{Y}_h) \otimes \mathcal{E}^1$ whereas τ_-, $\theta_- : \mathcal{E}_k(\mathcal{Y}_h) \to \mathcal{W}_{k-1} \otimes \mathcal{E}^1$.

To continue, we note that

- To each filtered bundle $(E, \mathcal{W})$ one may associate a corresponding graded vector bundle $Gr^{\mathcal{W}}$ via the rule:

$$Gr_k^{\mathcal{W}} = \frac{\mathcal{W}_k}{\mathcal{W}_{k-1}}$$

- By virtue of the induced isomorphism $\mathcal{Y}_h : E \cong Gr^{\mathcal{W}}$, the choice of a hermitian metric h on a filtered bundle $(E, \mathcal{W})$ determines an corresponding hermitian metric $h_{Gr^{\mathcal{W}}}$ on $Gr^{\mathcal{W}}$.

Definition 3.7 *A hermitian metric h on a flat, filtered bundle $(E, \mathcal{W}, \nabla)$ is said to be graded-harmonic provided it is harmonic with respect to the flat connection d_h (i.e. the induced metric $h_{Gr^{\mathcal{W}}}$ on $Gr^{\mathcal{W}}$ is harmonic).*

In particular, the choice of a graded-harmonic metric h on a flat filtered bundle $(E, \mathcal{W}, \nabla)$ determines an associated decomposition

$$d_h = \bar{\theta}_0 + \bar{\partial}_0 + \partial_0 + \theta_0 \tag{3.8}$$

by application of Construction (3.2) to the pair (h, d_h).

Definition 3.9 *Let $(E, \mathcal{W}, \nabla)$ be a flat, filtered bundle. Then, a hermitian metric h on E is said to be filtered harmonic provided:*

(a) *h is graded-harmonic.*

(b) *The differential operator $\bar{\partial} + \theta$ on E obtained by setting*

$$\bar{\partial} = \bar{\partial}_0 + \tau_-, \qquad \theta = \theta_0 + \theta_-$$

defined by equations (3.6) *and* (3.8) *is of Higgs–type.*

To relate the preceding constructions to variations of graded-polarized mixed Hodge structure, we recall from §2 that such a variation $\mathcal{V}$ comes equipped with a canonical C^∞ bigrading

$$\mathcal{V} = \bigoplus_{p,q} \mathcal{I}^{p,q} \tag{3.10}$$

such that:

- For each index p, $\mathcal{F}^p = \oplus_{a \geq p} \mathcal{I}^{a,b}$.
- For each index k, $\mathcal{W}_k = \oplus_{a+b \leq k} \mathcal{I}^{a,b}$.
- For each bi-index (p,q), $\overline{\mathcal{I}^{p,q}} = \mathcal{I}^{q,p} \mod \bigoplus_{r<q,s<p} \mathcal{I}^{r,s}$.

In particular, on account of this bigrading, a variation of graded-polarized mixed Hodge structure $\mathcal{V}$ comes equipped with the following additional structures:

- A canonical grading $\mathcal{Y}$ of $\mathcal{W}$ which acts as multiplication by $p+q$ on $\mathcal{I}^{p,q}$.

- A canonical mixed Hodge metric h defined by pulling back the Hodge metric of $Gr^{\mathcal{W}}$ via the induced isomorphism $\mathcal{Y} : \mathcal{V} \cong Gr^{\mathcal{W}}$.

- A canonical Higgs bundle structure $\bar{\partial} + \theta$ (cf. [12]).

To explain the construction of the Higgs bundle structure $\bar{\partial} + \theta$ on the underlying C^∞ vector bundle of $\mathcal{V}$, we recall that a complex variation of Hodge structure ($\mathbb{C}$VHS) consists of a flat vector bundle (E, ∇) equipped with a C^∞-decomposition

$$E = \bigoplus_p E^p \tag{3.11}$$

which satisfies the horizontality condition

$$\nabla : \mathcal{E}^0(E^p) \to \mathcal{E}^{0,1}(E^{p+1}) \oplus \mathcal{E}^{0,1}(E^p) \oplus \mathcal{E}^{1,0}(E^p) \oplus \mathcal{E}^{1,0}(E^{p-1}) \tag{3.12}$$

and note that a parallel hermitian bilinear form S is then said to polarize such a variation E provided:

- The direct sum decomposition (3.11) is orthogonal with respect to S.

- The associated "Hodge metric"

  $$h(u,v) = (-1)^p S(u,v), \qquad u, v \in E^p$$

  is positive definite.

Moreover, as with variations of pure, polarized Hodge structure, the Hodge metric of a polarized $\mathbb{C}$VHS is harmonic.

Lemma 3.13 *Let E is a complex variation of Hodge structure and $\bar{\partial} + \theta$ be the differential operator on E obtained by decomposing the underlying flat connection*

$$\nabla = \bar{\theta} + \bar{\partial} + \partial + \theta$$

of E in accord with equation (3.12)*. Then, $(\bar{\partial} + \theta)^2 = 0$ and hence $\bar{\partial} + \theta$ is an operator of Higgs–type.*

Proof. One simply expands out the flatness condition $\nabla^2 = 0$ and taking note of the additional requirements imposed by equation (3.11). □

Theorem 3.14 *Let $\mathcal{V}$ be a variation of graded-polarized mixed Hodge structure. Then, the C^∞ decomposition*

$$\mathcal{V} = \bigoplus_p E^p, \qquad E^p = \bigoplus_q \mathcal{I}^{p,q}$$

defines a complex variation of (unpolarized) Hodge structure on the underlying flat bundle of $\mathcal{V}$.

Corollary. *A variation of graded-polarized mixed Hodge structure carries a canonical Higgs bundle structure $\bar{\partial} + \theta$.*

In order to obtain a partial converse of the Lemma (3.13), recall that a decomposition

$$E = \bigoplus_p E^p$$

of a Higgs bundle $(E, \bar{\partial} + \theta)$ into a sum of holomorphic subbundles is said to be a system of Hodge bundles if and only if

$$\theta : E^p \to E^{p-1} \otimes \Omega^1$$

Lemma 3.15 *A Higgs bundle $(E, \bar{\partial} + \theta)$ defined over a compact complex manifold X admits a decomposition into a system of Hodge bundles if and only if*

$$(E, \bar{\partial} + \theta) \cong (E, \bar{\partial} + \lambda\theta)$$

for each element $\lambda \in \mathbb{C}^*$.

Proof. If $(E, \bar{\partial} + \theta)$ admits a decomposition into a sum of Hodge bundles $E = \bigoplus_p E^p$ then the desired isomorphism $f : (E, \bar{\partial} + \theta) \cong (E, \bar{\partial} + \lambda\theta)$ may be obtained by setting $f = \lambda^p$ on E^p.

Conversely, given a Higgs bundle $(E, \bar{\partial}+\theta)$ which is a fixed point of the $\mathbb{C}^*$ action $(E, \bar{\partial}+\theta) \to (E, \bar{\partial}+\lambda\theta)$ one may obtained the desired decomposition of E into a system of Hodge bundles as follows: Let f be an isomorphism from $(E, \bar{\partial}+\theta)$ to $(E, \bar{\partial}+\lambda\theta)$ for some element $\lambda \in \mathbb{C}^*$ which is not a root of unity. Then, because f is holomorphic and X is compact, the characteristic polynomial of f is constant. Upon decomposing E into a sum of generalized eigenspaces, one then obtains the desired system of Hodge bundles. □

Corollary 3.16 (Lemma 4.2, [14]) *Let X be a compact Kähler manifold. Then, a semisimple flat bundle $E \to X$ underlies a polarized $\mathbb{C}$VHS if and only if the corresponding Higgs bundle $(E, \bar{\partial} + \theta)$ is a fixed point of the $\mathbb{C}^*$-action*

$$(E, \bar{\partial} + \theta) \mapsto (E, \bar{\partial} + \lambda\theta)$$

Moreover, there is a 1-1 correspondence between the set of possible complex variations of polarized Hodge structure on such a semisimple flat bundle E and the set of possible decompositions of $(E, \bar{\partial} + \theta)$ into a system of Hodge bundles.

Motivated by the computations of [12], we now make the following definition:

Definition 3.17 *A complex variation of mixed Hodge structure consists of a flat vector bundle (E, ∇) endowed with a smooth decomposition of the underlying C^∞-vector bundle into a sum of subbundles*

$$E = \bigoplus_{p,q} E^{p,q}$$

such that

(*a*) *For each index k, the sum $\mathcal{W}_k = \bigoplus_{p+q \le k} E^{p,q}$ is a flat subbundle of E.*

(*b*) *If $E^p = \bigoplus_q E^{p,q}$ then for each bi-index (p, q),*

$$\nabla : \mathcal{E}^0(E^{p,k-p}) \to \mathcal{E}^{0,1}(E^{p+1,k-p+1}) \oplus \mathcal{E}^{0,1}(E^p) \oplus \mathcal{E}^{1,0}(E^{p,k-p}) \oplus \mathcal{E}^{1,0}(E^{p-1})$$

Remark. *It is perhaps more natural to define a $\mathbb{C}$VMHS as consisting of a $\mathbb{C}$-local system $\mathcal{V}$ endowed with a triple of suitably opposed filtrations $(\mathcal{F}, \mathcal{W}, \bar{\mathcal{F}})$ which satisfy the requisite horizontality conditions. One then obtains a functor from this category to the category of $\mathbb{C}$VMHS defined above via the map*

$$(\mathcal{F}, \mathcal{W}, \bar{\mathcal{F}}) \mapsto \bigoplus_{p,q} \mathcal{I}^{p.q}$$

However, in the absence of additional data (such as a real structure for $\mathcal{V}$), one can only recover $\mathcal{F}$ and $\mathcal{W}$ from the $\mathcal{I}^{p,q}$'s, and not the full triple $(\mathcal{F}, \mathcal{W}, \bar{\mathcal{F}})$.

To define the notion of a complex variation of graded-polarized mixed Hodge structure, note that by virtue of conditions (a) and (b), a complex variation of mixed Hodge structure induces complex variations of pure Hodge structure on $Gr^{\mathcal{W}}$ via the rule

$$E^p Gr^{\mathcal{W}}_k = \frac{E^p \cap \mathcal{W}_k + \mathcal{W}_{k-1}}{\mathcal{W}_{k-1}}$$

Definition 3.18 *A complex variation of graded-polarized mixed Hodge structure consists of a complex variation of mixed Hodge structure together with a collection of parallel hermitian bilinear forms on $Gr^{\mathcal{W}}$ which polarize the induced variations.*

In particular, every complex variation of graded-polarized mixed Hodge structure carries a canonical mixed Hodge metric obtained by simply pulling back the Hodge metrics of $Gr^{\mathcal{W}}$ via the grading

$$E_k(\mathcal{Y}) = \bigoplus_{p+q=k} E^{p,q} \tag{3.19}$$

Theorem 3.20 *The Deligne–Hodge decomposition*

$$\mathcal{V} = \bigoplus_{p,q} \mathcal{I}^{p,q}$$

of a variation of graded-polarized mixed Hodge structure $\mathcal{V} \to S$ defines a complex variation of graded-polarized mixed Hodge structure with respect to the underlying flat structure of $\mathcal{V}$.

Proof. Since the weight filtration $\mathcal{W}$ of $\mathcal{V}$ is by definition flat, it will suffice to verify condition (b) of Definition (3.17) by simply computing the action of the Gauss–Manin connection ∇ of $\mathcal{V}$ upon a smooth section σ of $\mathcal{I}^{p,q}$ at an arbitrary point $s_0 \in S$.

Accordingly, we recall from §2 that $\mathcal{V}$ may be represented near s_0 by a holomorphic, horizontal map

$$F : \Delta^n \to \mathcal{M}$$

from the polydisk Δ^n into a suitable classifying space $\mathcal{M}$ upon selection a system of holomorphic local coordinates $(s_1, \dots, s_n)$ on S which vanish at s_0. In particular, if $F = F(0)$ and

$$q_F = \bigoplus_{r<0, r+s\leq 0} \mathfrak{g}^{r,s}$$

denotes the vector space complement to $Lie(G^F_{\mathbb{C}})$ in $Lie(G_{\mathbb{C}})$ constructed in §2 [cf. Theorem (2.5)], it then follows that there exists a neighborhood of zero in Δ^n over which we may write

$$F(s) = e^{\Gamma(s)}.F$$

relative to a unique q_F-valued holomorphic function $\Gamma(s)$ vanishing at the origin.

To continue our calculations, we recall [12] that for sufficiently small values of s, one may write

$$I^{p,q}_{(F(s),W)} = e^{\Gamma(s)}e^{-\phi(s)}.I^{p,q}_{(F,W)}$$

relative to a C^∞ function $\phi(s)$ which has a first order Taylor series expansion given by the the formula

$$\phi(s) = -\pi(\overline{L(s)})$$

where $\pi : Lie(G_{\mathbb{C}}) \to Lie(G^F_{\mathbb{C}})$ denotes projection with respect to the the decomposition

$$Lie(G_{\mathbb{C}}) = Lie(G^F_{\mathbb{C}}) \oplus q_F$$

and

$$L(s) = \sum_{j=1}^n \frac{\partial\Gamma(0)}{\partial s_j} s_j$$

denotes the linearization of $\Gamma(s)$ about $s = 0$.

To determine how ∇ acts upon a smooth section σ of $I^{p,q}_{(F(s),W)}$ observe that by virtue of the preceding remarks, we may write

$$\sigma(s) = e^{\Gamma(s)}e^{-\phi(s)}\tilde{\sigma}(s)$$

relative to a unique function $\tilde{\sigma}(s)$ taking values in the fixed vector space $I^{p,q}_{(F,W)}$. Consequently,

$$\begin{aligned} \nabla\sigma|_0 &= de^{\Gamma(s)}e^{-\phi(s)}\Big|_0 \sigma(0) + d\tilde{\sigma}|_0 \\ &= dL|_0\,\sigma(0) + d\pi(\bar{L})\big|_0\,\sigma(0) + d\tilde{\sigma}|_0 \end{aligned}$$

In particular, since

$$L(s) : \Delta^n \to \bigoplus_{s\leq 1} \mathfrak{g}^{-1,s}$$

on account of the horizontality of $F(s)$,

$$\nabla^{1,0}\sigma\big|_0 \in I^{p,q}_{(F,W)} + \sum_{\ell\geq 0} I^{p-1,q+1-\ell}_{(F,W)}, \qquad \nabla^{0,1}\sigma\big|_0 \in I^{p+1,q-1}_{(F,W)} + \sum_{\ell\geq 0} I^{p,q-\ell}_{(F,W)}$$

□

In order to state the next result, we note there is a natural functor from the category of $\mathbb{C}$VMHS to the category of flat, filtered bundles which operates by replacing the bigrading $E = \oplus_{p,q} E^{p,q}$ of a $\mathbb{C}$VMHS by the associated filtration $\mathcal{W}_k = \bigoplus_{p+q\leq k} E^{p,q}$.

Theorem 3.21 *The mixed Hodge metric h carried by a complex variation of graded-polarized mixed Hodge structure E is filtered harmonic.*

Proof. By definition, a complex variation of graded-polarized mixed Hodge structure consists of a complex variation of mixed Hodge structure together with a collection of parallel hermitian forms on $Gr^{\mathcal{W}}$ which polarize the induced complex variations. Accordingly, the associated decomposition

$$d_h = \bar{\theta}_0 + \bar{\partial}_0 + \partial_0 + \theta_0$$

given by equation (3.8) has the additional property that

$$\bar{\theta}_0 : E^{p,k-p} \to E^{p+1,k-p-1} \otimes \mathcal{E}^{0,1}, \qquad \bar{\partial}_0 : \mathcal{E}^0(E^{p,k-p}) \to \mathcal{E}^{0,1}(E^{p,k-p})$$
$$\theta_0 : E^{p,k-p} \to E^{p-1,k-p+1} \otimes \mathcal{E}^{1,0}, \qquad \partial_0 : \mathcal{E}^0(E^{p,k-p}) \to \mathcal{E}^{1,0}(E^{p,k-p})$$

On the other hand, by condition (b) of Definition (3.17), if

$$\nabla = d_h + \tau_- + \theta_-$$

denotes the decomposition of ∇ defined by equation (3.6), then

$$\tau_- : E^p \to E^p \otimes \mathcal{E}^{0,1}, \qquad \theta_- : E^p \to E^{p-1} \otimes \mathcal{E}^{1,0}$$

and hence the decomposition

$$\nabla = \bar{\theta}_0 + \bar{\partial} + \partial_0 + \theta$$

obtained by setting $\bar{\partial} = \partial_0 + \tau_-$ and $\theta = \theta_0 + \theta_-$ coincides with usual decomposition of E defined by equation (3.12) via the system of Hodge bundles $E^p = \oplus_q E^{p,q}$. In particular, $\bar{\partial} + \theta$ is an operator of Higgs–type. □

Moreover, in analogy with [14], we have the following correspondence between filtered harmonic metrics and complex variations of graded-polarized mixed Hodge structure:

Theorem 3.22 *Let X be a compact Kähler manifold and $E \to X$ a flat, filtered bundle. Then, a filtered harmonic metric h on E underlies a complex variation of graded-polarized mixed Hodge structure if and only if*

(*a*) $(E, \bar{\partial} + \theta)$ *is a fixed point of the* $\mathbb{C}^*$*-action*

$$(E, \bar{\partial} + \theta) \mapsto (E, \bar{\partial} + \lambda\theta)$$

(*b*) *The resulting decomposition of* E *into a system of Hodge bundles*

$$E = \bigoplus_p E^p$$

is preserved by the grading $\mathcal{Y}_h$.

Proof. According to Theorem (3.21), the mixed Hodge metric of a complex variation of graded-polarized mixed Hodge structure is filtered harmonic. Conversely, given a filtered harmonic metric for which the above two conditions hold one defines

$$E^{p,k-p} = E_k(\mathcal{Y}_h) \cap E^p \tag{3.23}$$

To prove the bigrading (3.23) does indeed define a complex variation of graded-polarized mixed Hodge structure for which the associated mixed Hodge metric equals the given hermitian metric h, we note that for each index k, the decomposition

$$E_k(\mathcal{Y}_h) = \bigoplus_p E^p_k, \qquad E^p_k = E^{p,k-p} \tag{3.24}$$

defines a system of Hodge bundles with respect to $\bar{\partial}_0 + \theta_0$. Indeed, this is automatic given conditions (a) and (b) since $\bar{\partial}_0 + \theta_0$ is just the component of $\bar{\partial} + \theta$ which preserves $\mathcal{Y}_h$. On the other hand, by definition, $\bar{\partial}_0 + \theta_0$ coincides with the Higgs bundle structure obtained by applying Construction (3.2) to the pair (h, d_h). Consequently, by Corollary (3.16), there exists a unique complex variation of polarized Hodge structure on $E_k(\mathcal{Y}_h)$ for which h is the Hodge metric, d_h is the flat connection and equation (3.24) represents the decomposition of $E_k(\mathcal{Y}_h)$ into a system of Hodge bundles. Therefore, the decomposition

$$d_h = \bar{\theta}_0 + \bar{\partial}_0 + \partial_0 + \theta_0$$

given by equation (3.8) has the additional property that

$$\bar{\theta}_0 : E^{p,k-p} \to E^{p+1,k-p-1} \otimes \mathcal{E}^{0,1}, \qquad \bar{\partial}_0 : \mathcal{E}^0(E^{p,k-p}) \to \mathcal{E}^{0,1}(E^{p,k-p})$$

$$\theta_0 : E^{p,k-p} \to E^{p-1,k-p+1} \otimes \mathcal{E}^{1,0}, \qquad \partial_0 : \mathcal{E}^0(E^{p,k-p}) \to \mathcal{E}^{1,0}(E^{p,k-p})$$

Returning to condition (a), it then follows that if

$$\nabla = d_h + \tau_- + \theta_-$$

denotes the decomposition of ∇ defined by equation (3.6) then

$$\tau_- : E^p \to E^p \otimes \mathcal{E}^{0,1}, \qquad \theta_- : E^p \to E^{p-1} \otimes \mathcal{E}^{1,0}$$

since $\tau_- = \bar{\partial} - \bar{\partial}_0$ and $\theta_- = \theta - \theta_0$ on account of the filtered harmonicity of h. Consequently,

$$\nabla : \mathcal{E}^0(E^{p,k-p}) \to \mathcal{E}^{0,1}(E^{p+1,k-p-1}) \oplus \mathcal{E}^{0,1}(E^p) \oplus \mathcal{E}^{1,0}(E^{p,k-p}) \oplus \mathcal{E}^{1,0}(E^{p-1})$$

$\square$

4 Admissibility Criteria

Let $\mathcal{V} \to S$ be a variation of graded-polarized mixed Hodge structure and $\bar{S}$ be a compactification for which the divisor $D = \bar{S} - S$ has at worst normal crossings. Then, in contrast to the pure case, the associated period map $\varphi : S \to \mathcal{M}/\Gamma$ may have irregular singularities along D, as can be seen by considering the simplest of Hodge–Tate variations (cf. §2).

To rectify this problem, let us return to our prototypical example

$$\mathcal{V} = R^k_{f*}(\mathbb{C}) \tag{4.1}$$

defined by a family of algebraic varieties $f : X \to \Delta^*$ with unipotent monodromy. Then, as discussed in [16], the limiting Hodge filtration F_∞ of $\mathcal{V}$ exists.

Thus, comparing the asymptotic behavior of (4.1) and (4.2), we see that a minimal condition required in order for an abstract variation $\mathcal{V} \to \Delta^*$ with unipotent monodromy to be akin to a geometric variation is that:

(i) The limiting Hodge filtration of $\mathcal{V}$ exists (in the sense of §2).

Moreover, as shown by Deligne using ℓ-adic techniques [4], the geometric variations (4.2) are subject to a subtle condition which greatly restricts their local monodromy. Namely, if W denotes the constant value of the weight filtration of $\mathcal{V}$, then:

(ii) The relative weight filtration ${}^rW = {}^rW(N, W)$ of the monodromy logarithm N of $\mathcal{V}$ exists.

Consequently, on the basis of such considerations, we shall adopt the terminology of [16] and call an abstract variation $\mathcal{V} \to \Delta^*$ with unipotent monodromy *admissible* provided it satisfies conditions (i) and (ii).

Theorem. (Deligne, [16]) *The limiting Hodge filtration of an admissible variation of graded-polarized mixed Hodge structure $\mathcal{V} \to \Delta^*$ with unipotent monodromy pairs with the relative weight filtration of $\mathcal{V}$ to define a mixed Hodge structure for which N is morphism of type $(-1,-1)$.*

To state main result of this section, let ${}^rY$ be a grading of the relative weight filtration ${}^rW = {}^rW(N,W)$ which is *compatible* with N and W in the following sense:

- ${}^rY$ preserves W.
- $[{}^rY, N] = -2N$.

Then, given any grading Y of W preserving ${}^rW$, we may construct an associate sl_2 representation

$$(N_0, {}^rY - Y, N_0^+) \tag{4.2}$$

on the underlying vector space of ${}^rW$ by decomposing N as

$$N = N_0 + N_{-1} + N_{-2} + \cdots \tag{4.3}$$

according to the eigenvalues of $ad\,Y$ and applying Theorem (2.15) to the pair

$$(N_0, {}^rY - Y)$$

Theorem 4.5 (Deligne, [5]) *Let ${}^rY$ be a grading of ${}^rW(N,W)$ which is compatible with N and W. Then, there exists a unique grading Y of W such that Y preserves ${}^rW$ and*

$$[N - N_0, N_0^+] = 0 \tag{4.6}$$

Proof. We begin by selecting a grading Y_0 of W which preserves ${}^rW$, and recalling that by [3], the group

$$G^\circ = \{\, g \in GL(V)^{{}^rY} \mid (g-1)(W_k) \subseteq W_{k-1} \,\}$$

acts simply transitively on the set of all such gradings $\mathcal{Y}({}^rY, W)$. Next, to compute how this transitive action changes the associated sl_2 representations (4.3), write

$$N = N_0 + N_{-1} + N_{-2} + \cdots$$

relative to $Y \in \mathcal{Y}({}^rY, W)$, and observe that

$$g \in G^\circ \implies Gr(Ad(g)\, N_0) = Gr(N) \tag{4.7}$$

Consequently, by (4.7), the decomposition

$$N = N_0^* + N_{-1}^* + N_{-2}^* + \cdots$$

of N relative to $g.Y$ must have $N_0^* = Ad(g)\, N_0$, hence

$$(N_0, {}^rY - Y, N_0^+) \xrightarrow{g} Ad(g)(N_0, {}^rY - Y, N_0^+) \tag{4.8}$$

To prove the existence of a grading $Y \in \mathcal{Y}({}^rY, W)$ which satisfies (4.6), we proceed by induction and construct a sequence of gradings $Y_0, Y_1, \ldots, Y_n$ in $\mathcal{Y}({}^rY, W)$ terminating at Y, by the requirement that

$$[N - N_0, N_0^+] : W_j \to W_{j-\ell-1} \tag{4.9}$$

upon decomposing N relative to $ad(Y_\ell)$. To check the validity of this algorithm, let us suppose the gradings $Y_0, Y_1, \ldots, Y_{k-1}$ have been constructed, and write

$$Y_k = g.Y_{k-1}, \qquad g \in G^\circ$$

Then, by virtue of equation (4.8), we can reformulate condition (4.9) as the requirement that

$$[Ad(g^{-1})N - N_0, N_0^+] : W_j \to W_{j-k-1} \tag{4.10}$$

with

$$N = N_0 + N_{-1} + N_{-2} + \cdots$$

denoting the decomposition of N relative to $ad(Y_{k-1})$.

Now, by virtue of the fact that

$$[N - N_0, N_0^+] : W_j \to W_{j-k}$$

(upon decomposing N relative to $ad(Y_{k-1})$), it is natural to assume that our element g is of the form

$$g = 1 + \gamma_{-k} + \gamma_{-k-1} + \cdots$$

(again, relative to the eigenvalues of $ad(Y_{k-1})$). Imposing condition (4.10), it follows that we may take g to be of this form if and only there exists a solution γ_{-k} of the equation:

$$[N_{-k} + [N_0, \gamma_{-k}], N_0^+] = 0 \tag{4.11}$$

To find such an element γ_{-k}, one simply observes that

$$End(V) = \mathrm{Im}(ad\, N_0) \oplus \ker(adN_0^+)$$

To prove the grading Y so constructed is unique, suppose that Y' is another such grading and write $Y' = g.Y$ with

$$g = 1 + \gamma_{-k} + \gamma_{-k-1} + \cdots \in G^\circ, \qquad \gamma_{-k} \neq 0$$

relative to $ad(Y)$. Therefore, upon applying $Ad(g^{-1})$ to both sides of equation (4.8), we see that

$$[Ad(g^{-1})N - N_0, N_0^+] = 0$$

and hence, we also must have

$$[N_{-k} + [N_0, \gamma_{-k}], N_0^+] = [[N_0, \gamma_{-k}], N_0^+] = 0 \tag{4.12}$$

since $g = 1 + \gamma_{-k} + \gamma_{-k-1} + \cdots$.

In addition, by combining the observation that ${}^rY$ preserves the eigenspaces of Y with the fact that $g \in G^\circ$ fixes ${}^rY$, it follows that we must also have

$$[\gamma_{-k}, {}^rY] = 0$$

Thus, γ_{-k} is a solution to (4.12) which is of weight $k > 0$ relative to the representation $(ad(N_0), ad({}^rY - Y), ad(N_0^+))$. By standard sl_2 theory, we therefore have $\gamma_{-k} = 0$, contradicting our our assumption that $g \neq 1$. $\square$

Remark. *Deligne [5]:*

- *Since* $[N - N_0, N_0^+] = 0$, *each non-zero term* N_{-k} *with* $k > 0$ *appearing in the decomposition* (4.4) *defines an irreducible representation of* $sl_2(\mathbb{C})$ *of highest weight* $k-2$ *via the adjoint action of* $(N_0, {}^rY - Y, N_0^+)$. *In particular,* $N_{-1} = 0$.

- *The construction of Theorem* (4.5) *is compatible with tensor products, directs sums and duals.*

- *The construction is functorial with respect to morphisms.*
  Sketch of Proof. Let $f = \sum_{k\geq 0} f_{-k}$ be the decomposition of our morphism $f : V_1 \to V_2$ relative to the induced grading of $Hom(V_1, V_2)$. Then, in analogy with the proof of Theorem (4.5), direct computation shows that relative to the induced representation (N_0, H, N_0^+) on $Hom(V_1, V_2)$ we must have both

  $$[N_0^+, [N_0, f_{-k}]] = 0$$

  and $H(f_{-k}) = kf_{-k}$.

To relate this construction of Deligne to admissible variations, let us suppose we have such a variation $\mathcal{V} \to \Delta^*$ with limiting mixed Hodge structure $(F_\infty, {}^rW)$. Then, because N is a morphism of $(F_\infty, {}^rW)$ of type $(-1,-1)$, we may apply Theorem (4.5) with

$$^rY = Y_{(F_\infty, {}^rW)} \tag{4.13}$$

to obtain an associate grading:

$$Y = Y(F_\infty, W, N)$$

Thus, motivated by these observations, let us call a triple (F, W, N) *admissible* provided:

(a) The relative weight filtration ${}^rW = {}^rW(N, W)$ exists.

(b) The pair $(F, {}^rW)$ induces a mixed Hodge structure on each W_k, relative to which N is $(-1,-1)$-morphism.

Then, mutatis mutandis, we obtain an associate grading

$$Y = Y(F, W, N) \tag{4.14}$$

for each admissible triple (F, W, N).

To study the dependence of the grading (4.15) upon the filtration F, recall that via the splitting operation described in [3], we can pass from an arbitrary mixed Hodge structure $(F, {}^rW)$ to a mixed Hodge structure $(\hat{F}, {}^rW) = (e^{-i\delta}.F, {}^rW)$ split over $\mathbb{R}$. Moreover, a moment of thought shows

$$e^{-i\delta} : ({}^rY, F, {}^rW) \to ({}^r\hat{Y}, N, W)$$

to be a morphism of admissible triples, hence

$$Y(F, W, N) = e^{i\delta}.Y(\hat{F}, W, N) \tag{4.15}$$

by the functoriality of Theorem (4.5).

To understand the power of this reduction to the split over $\mathbb{R}$ case, observe that every mixed Hodge structure which splits over $\mathbb{R}$ and has N as a $(-1,-1)$-morphism can be built up from the following two sub-cases:

(1) $V = \bigoplus I^{p,p}$ and $N(I^{p,p}) \subseteq I^{p-1,p-1}$.

(2) $V = I^{p,0} \bigoplus I^{0,p}$ and $N = 0$.

via the operations of tensor product and direct sums. In particular, because Theorem (4.5) is compatible with these operations, we obtain the following result:

Theorem 4.17 *Let (F, W, N) be an admissible triple. Then, the grading $Y = Y(F, W, N)$ preserves the filtration F.*

Proof. By virtue of the previous remarks, it will suffice to check the two sub-cases enumerated above. To verify the assertion for case (1), note that $I^{p,p}$ is the weight 2p eigenspace of ${}^{r}Y$, and recall that by definition we must have

$$F^p = \bigoplus_{a \geq p} I^{p,p} \tag{4.16}$$

Consequently, the commutativity relation $[{}^{r}Y, Y] = 0$ implies that

$$Y(I^{p,p}) \subseteq I^{p,p}$$

and hence Y preserves F by equation (4.18).

Regarding the second case, observe that because $N = 0$, W must be the trivial filtration

$$0 = W_{2p-1} \subset W_{2p} = V$$

in order for the relative weight filtration ${}^{r}W = {}^{r}W(N, W)$ to exist. Thus, Y must be the trivial grading on V of weight $2p$. $\square$

Returning now to the work of Schmid, let us recall the following basic result concerning the monodromy weight filtration [13]:

Lemma 4.19 *Let V be a finite dimensional $\mathbb{C}$-vector space endowed with a non-degenerate bilinear form Q, and suppose N is a nilpotent endomorphism of V which acts by infinitesimal isometries of Q. Then, any semisimple endomorphism H of V which satisfies $[H, N] = -2N$ is also an infinitesimal isometry of Q.*

Proof. There are two key steps:

(1) Show the monodromy weight filtration

$$0 \neq W_{-\ell}(N) \subseteq \dots W_{\ell}(N) = V \tag{4.20}$$

is self-dual with respect to Q, i.e. $W_j(N) = W_{-j-1}(N)^{\perp}$.

(2) Via semisimplicity, assume the pair (N, H) defines an sl_2 representation $\{e_k, e_{k-2}, \dots, e_{2-k}, e_{-k}\}$ of highest weight k. Imposing self-duality, it then follows that $Q(e_i, e_j) = 0$ unless $i + j = 0$, hence H is an infinitesimal isometry of Q.

$\square$

Corollary 4.20 *Let $\mathcal{V} \to \Delta^*$ be an admissible variation with limiting data (F_∞, W, N) and graded-polarizations $\{\mathcal{S}_k\}$. Then, the semi-simple element ${}^{r}Y - Y$ constructed by Theorem* (4.5) *acts on Gr^W by infinitesimal isometries.*

5 The Nilpotent Orbit Theorem

In this section, we state and prove a precise analog of Schmid's Nilpotent Orbit Theorem for admissible variations of graded-polarized mixed Hodge structure $\mathcal{V} \to \Delta^*$ with unipotent monodromy.

To this end, we recall from §2 that the period map

$$\varphi : \Delta^* \to \mathcal{M}/\Gamma$$

of such a variation lifts to a holomorphic, horizontal map $F(z) : U \to \mathcal{M}$ which makes the following diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{F} & \mathcal{M} \\ {\scriptstyle s=\exp(2\pi i z)}\downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\phi} & \mathcal{M}/\Gamma \end{array}$$

In particular, as noted in §2, the map $F(z)$ then descends to an "untwisted period map"

$$\psi(s) : \Delta^* \to \check{\mathcal{M}}$$

on account of the quasi-periodicity condition $F(z+1) = e^N.F(z)$.

Theorem 5.1 [Nilpotent Orbit Theorem] *Let $\psi(s)$ be the untwisted period map of an admissible variation of graded-polarized mixed Hodge structure $\mathcal{V} \to \Delta^*$ with unipotent monodromy. Then,*

(i) The limiting Hodge filtration $F_\infty = \lim_{s\to 0} \psi(s)$ of $\mathcal{V}$ exists, and is an element of $\check{\mathcal{M}}$.

(ii) The nilpotent orbit $F_{nilp}(z) = \exp(zN).F_\infty$ extends to a holomorphic, horizontal map $\mathbb{C} \to \check{\mathcal{M}}$. Moreover, there exists $\alpha > 0$ such that $F_{nilp}(z) \in \mathcal{M}$ whenever $\operatorname{Im}(z) > \alpha$.

(iii) Let $d_{\mathcal{M}}$ denote the distance function determined by the mixed Hodge metric h. Then, there exists constants K and β such that:

$$d_{\mathcal{M}}(F(z), F_{nilp}(z)) \leq K \operatorname{Im}(z)^\beta \exp(-2\pi \operatorname{Im}(z))$$

for all $z \in U$ with $\operatorname{Im}(z)$ sufficiently large.

Regarding the proof of Theorem (5.1), observe that part (*i*) is a direct consequence of the admissibility of $\mathcal{V}$ (cf. §2). Likewise, part (*ii*) is a direct consequence of Schmid's Nilpotent Orbit Theorem, applied to the variations of pure, polarized Hodge structure carried by Gr^W.

To prove part (*iii*), recall from §2 that $\check{\mathcal{M}}$ is a complex manifold upon which the Lie group

$$G_{\mathbb{C}} = \{\, g \in GL(V)^W \mid Gr(g) \in Aut(\mathcal{S}, \mathbb{C}) \,\}$$

acts transitively. Therefore, given any element $F \in \check{\mathcal{M}}$ and a vector space decomposition

$$Lie(G_{\mathbb{C}}) = Lie(G_{\mathbb{C}}^F) \oplus q \tag{5.2}$$

the map $u \in q \mapsto e^u.F$ will be a biholomorphism from a neighborhood of zero in q onto a neighborhood of F in $\check{\mathcal{M}}$. In particular, upon setting $F = F_\infty$, and shrinking Δ as necessary, we see that each choice of decomposition (5.2) determines a corresponding holomorphic map

$$\Gamma(s) : \Delta \to q, \qquad \Gamma(0) = 0$$

via the rule:

$$e^{\Gamma(s)}.F_\infty = \psi(s) \tag{5.3}$$

To make a choice of decomposition (5.2), we shall follow the methods of [12] and use the limiting mixed Hodge structure of $\mathcal{V}$ to define a grading of $Lie(G_{\mathbb{C}})$:

Lemma 5.4 *The limiting mixed Hodge structure $(F_{\infty}, {}^rW)$ of an admissible variation of graded-polarized mixed Hodge structure defines a canonical, graded, nilpotent Lie algebra*

$$q_{\infty} = \bigoplus_{a<0} \wp_a \tag{5.5}$$

with the following additional property: As complex vector spaces,

$$Lie(G_{\mathbb{C}}) = Lie(G_{\mathbb{C}}^{F_{\infty}}) \oplus q_{\infty} \tag{5.6}$$

Proof. The details may be found in §6 of [12]. However, the idea of the proof is relatively simple: An element $\alpha \in Lie(G_{\mathbb{C}})$ belongs to the subspace $\wp_a$ if and only if

$$\alpha : I^{p,q}_{(F_{\infty},{}^rW)} \to \bigoplus_b I^{p+a,b}_{(F_{\infty},{}^rW)}$$

To verify the decomposition (5.6), we observe that

(1) As a vector space, $Lie(G_{\mathbb{C}}) = \oplus_a \wp_a$.

(2) By definition, $F^p_{\infty} = \oplus_{a\geq p} I^{a,b}_{(F_{\infty},{}^rW)}$ and hence $Lie(G_{\mathbb{C}}^{F_{\infty}}) = \bigoplus_{a\geq 0} \wp$.

□

To establish the distance estimate (iii), let us now record the following result:

Lemma 5.7 *Let Y be a grading of W which is defined over $\mathbb{R}$ and y be a positive real number. Then,*

$$\alpha \in \mathbb{R} \implies y^{\alpha Y} : \mathcal{M} \to \mathcal{M}$$

Moreover, if $\alpha < 0$, $d_{\mathcal{M}}$ denotes the Riemann distance on $\mathcal{M}$ defined by the mixed Hodge metric (cf. §2) and L denotes the length of W then:

$$F_1, F_2 \in \mathcal{M} \implies d_{\mathcal{M}}(y^{\alpha Y}.F_1, y^{\alpha Y}.F_2) \leq y^{-\alpha(L-1)} d_{\mathcal{M}}(F_1, F_2) \tag{5.8}$$

Proof. To prove that $y^{\alpha Y} : \mathcal{M} \to \mathcal{M}$, one simply checks that $y^{\alpha Y}$ acts as scalar multiplication by $y^{\alpha k}$ on Gr^W_k, and hence $(y^{\alpha Y}.F)Gr^W_k = FGr^W_k$.

To verify equation (5.8), we note that if Y_F, $F \in \mathcal{M}$ denotes the grading of W which acts as multiplication by $(p+q)$ on $I^{p,q}_{(F,W)}$ then

$$Y_{y^{\alpha Y}.F} = Ad(y^{\alpha Y}) Y_F$$

since Y is defined over $\mathbb{R}$, and hence

$$v \in E_k(Y_F) \implies ||y^{\alpha Y} v||_{y^{\alpha Y}.F} = y^{\alpha k}||v||_F$$

Upon transferring these computations to the induced metrics on $Lie(G_{\mathbb{C}})$, it then follows that

$$T \in E_{-\ell}(ad\, Y_F) \implies ||Ad(y^{\alpha Y})T||_{y^{\alpha Y}.F} = y^{-\alpha\ell}||T||_F$$

In particular, $||Ad(y^{\alpha Y})T||_{y^{\alpha Y}.F} \leq y^{-\alpha(L-1)}||T||_F$ since $\alpha < 0$, and hence

$$d_{\mathcal{M}}(y^{\alpha Y}.F_1, y^{\alpha Y}.F_2) \leq y^{-\alpha(L-1)} d_{\mathcal{M}}(F_1, F_2)$$

$\square$

Remark. *In connection with the proof of Theorem* (5.9)*, we note the following result: Let* (M, g) *be a Riemannian manifold which is an open subset of a manifold* $\check{M}$ *upon which a Lie group* $\mathcal{G}$ *acts transitively, and* $|\cdot|$ *be a norm on* $Lie(\mathcal{G})$*. Then given any point* $F_0 \in M$ *there exists a neighborhood* S *of* F_0 *in* M*, a neighborhood* U *of zero in* $Lie(\mathcal{G})$ *and a constant* $K > 0$ *such that*

$$u \in U, F \in S \implies e^u.F \in M \quad \textit{and} \quad d_M(e^u.F, F) < K|u|$$

Theorem 5.9 **[Distance Estimate]** *Let* $F(z) : U \to \mathcal{M}$ *be a lifting of the period map* φ *of an admissible variation of graded-polarized mixed Hodge structure* $\mathcal{V} \to \Delta^*$ *with unipotent monodromy logarithm* N *and limiting mixed Hodge structure* $(F_\infty, {}^rW)$*. Then, given any* $G_{\mathbb{R}}$ *invariant metric on* $\mathcal{M}$ *which obeys* (5.8)*, there exists constants* K *and* β *such that*

$$Im(z) >> 0 \implies d_{\mathcal{M}}(F(z), e^{zN}.F_\infty) < K y^\beta e^{-2\pi y}$$

Proof. For simplicity of exposition, we shall first prove the result under the additional assumption that our limiting mixed Hodge structure $(F_\infty, {}^rW)$ is split over $\mathbb{R}$. Having made this assumption, it then follows from the work of §4 that:

(1) The associate gradings ${}^rY = Y_{(F_\infty, {}^rW)}$ and $Y = Y(F_\infty, W, N)$ are defined over $\mathbb{R}$.

(2) The endomorphism ${}^rY - Y$ is an element $Lie(G_{\mathbb{R}})$.

(3) The filtration $F_0 = e^{iN}.F_\infty$ is an element of $\mathcal{M}$. [This assertion is a consequence of Schmid's SL_2 Orbit Theorem, see [13] for details.]

Now, as discussed in §4, the fact that N is a $(-1,-1)$-morphism of the limiting mixed Hodge structure $(F_\infty, {}^rW)$ implies that

$$[{}^rY, N] = -2N$$

and hence

$$e^{iyN} = y^{-\frac{1}{2}{}^rY} e^{iN} y^{\frac{1}{2}{}^rY}$$

Consequently, because ${}^rY$ preserves F_∞,

$$e^{iyN}.F_\infty = y^{-\frac{1}{2}{}^rY} e^{iN} y^{\frac{1}{2}{}^rY} = y^{-\frac{1}{2}{}^rY} e^{iN}.F_\infty = y^{-\frac{1}{2}{}^rY}.F_0$$

Next, we note that by equation (5.3) and Lemma (5.4)

$$F(z) = e^{zN}.\psi(s) = e^{zN} e^{\Gamma(s)}.F_\infty$$

and hence

$$\begin{aligned} d_{\mathcal{M}}(e^{zN}e^{\Gamma(s)}.F_\infty, e^{zN}.F_\infty) &= d_{\mathcal{M}}(e^{iyN}e^{\Gamma(s)}.F_\infty, e^{iyN}.F_\infty) \\ &= d_{\mathcal{M}}(y^{-\frac{1}{2}{}^rY}e^{iN}y^{\frac{1}{2}{}^rY}e^{\Gamma(s)}.F_\infty, y^{-\frac{1}{2}{}^rY}e^{iN}y^{\frac{1}{2}{}^rY}.F_\infty) \end{aligned}$$

In particular, upon setting

$$e^{\tilde{\Gamma}(z)} = Ad(e^{iN})Ad(y^{\frac{1}{2}{}^rY})e^{\Gamma(s)}$$

and recalling that ${}^rY$ preserves F_∞, it follows that

$$d_{\mathcal{M}}(e^{zN}e^{\Gamma(s)}.F_\infty, e^{zN}.F_\infty) = d_{\mathcal{M}}(y^{-\frac{1}{2}{}^rY}e^{\tilde{\Gamma}(z)}.F_0, y^{-\frac{1}{2}{}^rY}.F_0) \tag{5.10}$$

In addition, because $[{}^rY, Y] = 0$,

$$y^{-\frac{1}{2}{}^rY} = y^{-\frac{1}{2}({}^rY - Y)}y^{-\frac{1}{2}Y}$$

and hence

$$d_{\mathcal{M}}(y^{-\frac{1}{2}{}^rY}e^{\tilde{\Gamma}(z)}.F_0, y^{-\frac{1}{2}{}^rY}.F_0) = d_{\mathcal{M}}(y^{-\frac{1}{2}Y}e^{\tilde{\Gamma}(z)}.F_0, y^{-\frac{1}{2}Y}.F_0) \tag{5.11}$$

since ${}^rY - Y$ is an element of $Lie(G_{\mathbb{R}})$. Therefore, by equation (5.8),

$$d_{\mathcal{M}}(e^{zN}e^{\Gamma(s)}.F_\infty, e^{zN}.F_\infty) \leq y^{\frac{1}{2}(L-1)} d_{\mathcal{M}}(e^{\tilde{\Gamma}(z)}.F_0, F_0)$$

Consequently, by the remark which follows Lemma (5.7), we have:

$$d_{\mathcal{M}}(e^{zN}e^{\Gamma(s)}.F_\infty, e^{zN}.F_\infty) < K|\tilde{\Gamma}(z)|\mathrm{Im}(z)^{\frac{1}{2}(L-1)}$$

for some $K > 0$ and norm any fixed norm $|\cdot|$ on $gl(V)$.

To further analyze $\tilde{\Gamma}(z)$, decompose $\tilde{\Gamma}(z)$ according to the eigenvalues of $ad\,{}^rY$

$$\Gamma = \sum_\ell \Gamma_{[\ell]}, \qquad [{}^rY, \Gamma_{[\ell]}] = \ell\Gamma_{[\ell]}$$

Then

$$\begin{aligned}\tilde{\Gamma}(z) &= e^{i\,ad\,N}y^{\frac{1}{2}ad\,{}^rY}\Gamma(s) = e^{i\,ad\,N}y^{\frac{1}{2}ad\,{}^rY}\sum_\ell \Gamma_{[\ell]}(s) \\ &= \sum_\ell y^{\frac{\ell}{2}}e^{i\,ad\,N}\Gamma_{[\ell]}(s)\end{aligned}$$

since $y^{\frac{1}{2}ad^rY}\Gamma_{[\ell]} = y^{\frac{\ell}{2}}\Gamma_{[\ell]}$. Therefore, denoting the maximal eigenvalue of $ad\,{}^rY$ on q_∞ by c, we have

$$|\tilde{\Gamma}(z)| \leq O(y^{c/2}e^{-2\pi y})$$

because $\Gamma(s)$ is a holomorphic function of $s = e^{2\pi iz}$ vanishing at zero.

In summary, we have proven that whenever the limiting mixed Hodge structure $(F_\infty, {}^rW)$ of our admissible variation $\mathcal{V}$ is split over $\mathbb{R}$ and $Im(z)$ is sufficiently large, the following distance estimate holds:

$$d_{\mathcal{M}}(e^{zN}e^{\Gamma(s)}.F_\infty, e^{zN}.F_\infty) \leq Ky^\beta e^{-2\pi y}, \qquad \beta = \frac{1}{2}(c + L - 1)$$

If the limiting mixed Hodge structure $(F_\infty, {}^rW)$ is not split over $\mathbb{R}$, we may obtain the same distance estimate by first applying Deligne's δ splitting:

$$F_\infty = e^{i\delta}.\hat{F}_\infty$$

and then proceeding as above.

More precisely, let ${}^rY, Y$ denote the gradings determined by $(\hat{F}_\infty, {}^rW)$ and N. Then, equation (5.10) becomes

$$d_{\mathcal{M}}(e^{zN}e^{\Gamma(s)}.F_\infty, e^{zN}.F_\infty) = d_{\mathcal{M}}(y^{-\frac{1}{2}Y}e^{\tilde{\Gamma}(z)}.F_0(y), y^{-\frac{1}{2}Y}.F_0(y))$$

with

$$F_0(y) = e^{i\delta(y)}e^{iN}.\hat{F}_\infty, \qquad \delta(y) = y^{\frac{1}{2}ad\,{}^rY}\delta$$

and

$$\delta \in \Lambda^{-1,-1}_{(F_\infty, {}^rW)} \implies \delta(y) \sim O(y^{-1})$$

Thus, applying equation (5.8) and the remark the follows the proof of Lemma (5.7), we have:

$$d_{\mathcal{M}}(e^{zN}e^{\Gamma(s)}.F_\infty, e^{zN}.F_\infty) \leq Ky^\beta e^{-2\pi y}$$

for $Im(z)$ sufficiently large. $\square$

In regards to analogs of Schmid's SL_2 Orbit Theorem for admissible variations $\mathcal{V} \to \Delta^*$ of graded-polarized mixed Hodge structure, the following result suggests that as soon as the weight filtration of $\mathcal{V}$ has length $L \geq 3$, the resulting nilpotent orbit $e^{zN}.F_\infty$ need not admit an approximation by an auxiliary orbit $e^{zN}.\hat{F}_\infty$ for which the limiting mixed Hodge structure $(\hat{F}_\infty, {}^rW)$ is split over $\mathbb{R}$:

Theorem 5.11 *Let $(F_\infty, {}^rW)$ be the limiting mixed Hodge structure associated to an admissible variation of graded-polarized mixed Hodge structure with unipotent monodromy and $(\hat{F}_\infty, {}^rW)$ be a mixed Hodge structure of the form*

$$(\hat{F}_\infty, {}^rW) = (e^{-\sigma}.F_\infty, {}^rW), \qquad \sigma \in \ker(ad\, N) \cap \Lambda^{-1,-1}_{(F_\infty, {}^rW)}$$

which is split over $\mathbb{R}$. Then, there exist a positive constant K such that

$$d_{\mathcal{M}}(e^{zN}.\hat{F}_\infty, e^{zN}.F_\infty) \leq K\, \mathrm{Im}(z)^{\frac{L-3}{2}}$$

for all $z \in U$ with $\mathrm{Im}(z)$ sufficiently large.

Proof. Let ${}^rY$ and Y denote the gradings associated to the split mixed Hodge structure $(\hat{F}_\infty, {}^rW)$ via the methods of §4. Then, a quick review of the proof of Theorem (5.9) shows that upon setting

- $F_0 = e^{iN}.\hat{F}_\infty \in \mathcal{M}$.
- $\sigma(y) = (y^{\frac{1}{2} ad\, {}^rY})(\sigma)$.

we have both $e^{zN}.\hat{F}_\infty = e^{xN}y^{-\frac{1}{2}{}^rY}.F_0$ and $e^{zN}.F_\infty = e^{xN}y^{-\frac{1}{2}{}^rY}e^{\sigma(y)}.F_0$

In particular, since ${}^rY - Y \in Lie(G_{\mathbb{R}})$ and Y is a grading of W which is defined over $\mathbb{R}$:

$$\begin{aligned}
d_{\mathcal{M}}(e^{zN}.\hat{F}_\infty, e^{zN}.F_\infty) &= d_{\mathcal{M}}(e^{xN}y^{-\frac{1}{2}{}^rY}.F_0, e^{xN}y^{-\frac{1}{2}{}^rY}e^{\sigma(y)}.F_0) \\
&= d_{\mathcal{M}}(y^{-\frac{1}{2}({}^rY-Y)}y^{-\frac{1}{2}Y}.F_0, y^{-\frac{1}{2}({}^rY-Y)}y^{-\frac{1}{2}Y}e^{\sigma(y)}.F_0) \\
&= d_{\mathcal{M}}(y^{-\frac{1}{2}Y}.F_0, y^{-\frac{1}{2}Y}e^{\sigma(y)}.F_0) \\
&\leq y^{\frac{1}{2}(L-1)}d_{\mathcal{M}}(F_0, e^{\sigma(y)}.F_0)
\end{aligned}$$

Moreover, $\sigma \in \Lambda^{-1,-1}_{(F_\infty, {}^rW)} \implies \sigma(y) \sim O(y^{-1})$ and hence

$$d_{\mathcal{M}}(F_0, e^{\sigma(y)}.F_0) \leq K y^{-1}$$

Therefore,

$$d_{\mathcal{M}}(e^{zN}.\hat{F}_\infty, e^{zN}.F_\infty) \leq K\,\mathrm{Im}(z)^{\frac{L-3}{2}}$$

$\square$

To verify that the distance estimate of Theorem (5.11) is sharp in the case where $L = 3$, let $\mathcal{M} \cong \mathbb{C}$ denote the classifying space of Hodge–Tate structures constructed in Example (2.7), and N be the nilpotent endomorphism of $V = \mathrm{span}(e_2, e_0)$ defined by the rule

$$N(e_2) = e_0, \qquad N(e_0) = 0$$

Then, a short calculation shows that

- ${}^rW = {}^rW(N, W)$ exists and coincides with W.
- At each point $F \in \mathcal{M}$, $Lie(G_{\mathbb{C}}) = \mathfrak{g}^{-1,-1}_{(F,W)} = \mathrm{span}_{\mathbb{C}}(N)$.

In particular, given any point $F_\infty \in \mathcal{M}$, the map

$$z \mapsto e^{zN}.F_\infty$$

is an admissible nilpotent orbit. Moreover, since $\mathcal{M}$ is Hodge–Tate, the resulting mixed Hodge metric on $\mathcal{M}$ is invariant under left translation by $e^{\lambda N}$ for all $\lambda \in \mathbb{C}$. Consequently, upon setting

$$F_\infty = e^{\sigma}.\hat{F}_\infty$$

for some element $\sigma \in \Lambda^{-1,-1}_{(F,W)}$ and some point $\hat{F}_\infty \in \mathcal{M}_{\mathbb{R}}$, it then follows that

$$d_{\mathcal{M}}(e^{zN}.\hat{F}_\infty, e^{zN}.F_\infty) = d_{\mathcal{M}}(e^{zN}.\hat{F}_\infty, e^{zN}e^{\sigma}.\hat{F}_\infty) = d_{\mathcal{M}}(\hat{F}_\infty, e^{\sigma}.\hat{F}_\infty)$$

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